Small Instanton Transitions Between the Gauge and Gravitational Sectors

Callum Brodie

Based on work with
Lara Anderson and **James Gray**
(first paper soon to appear)

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VIRGINIA TECH

NSF

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Topological transitions between CY3s well-studied

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in anomaly-consistent way

(And how to absorb back to get the two bundles)

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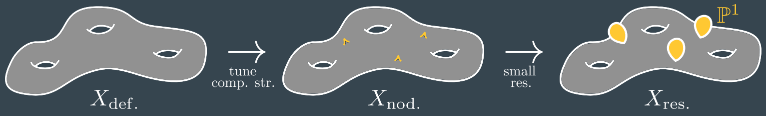
See James’s talk from Monday for:

- How following picture is inspired by target space duality
- More details on the pure bundle picture

The conifold transition



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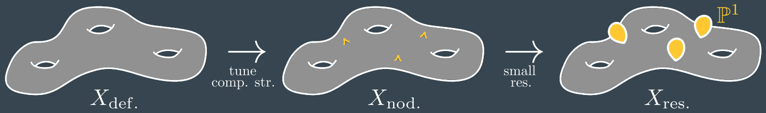

Importantly:

- Changes $h^{1,1}$ and $h^{2,1}$ (and so number of geometrical moduli)
- Changes c_2 (morally '$c_2(X_{\text{def.}}) = c_2(X_{\text{res.}}) + [\mathbb{P}^1\text{s}]$')

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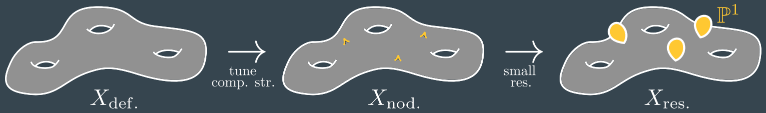

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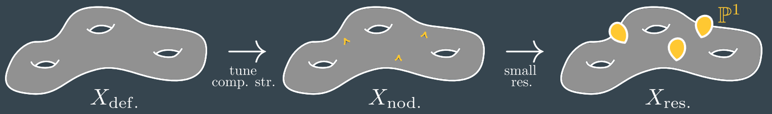

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Example (which we will use repeatedly):

$$X_{\text{def.}} \quad \to \quad X_{\text{nod.}} \quad \to \quad X_{\text{res.}}$$

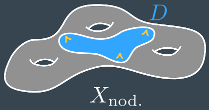
$$\left[\ \mathbb{P}^4 \ \middle|\ 5\ \right]^{1,101} \qquad \det\begin{pmatrix} l_1 & l_2 \\ q_1 & q_2 \end{pmatrix} = 0 \qquad \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^4 & 1 & 4 \end{array}\right]^{2,86}$$

Divisors associated to the transition

Important fact for this story:

Special divisors appear in nodal limit (Weil but non-Cartier)

(Captures appearance of new divisors as $h^{1,1}(X_{\text{def.}}) \to h^{1,1}(X_{\text{res.}})$)



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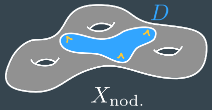


Easy to describe:

Generally for '$\mathbb{P}^n$-splits': Image of hyperplane $\{x_i = 0\}$ of $\mathbb{P}^n[x]$

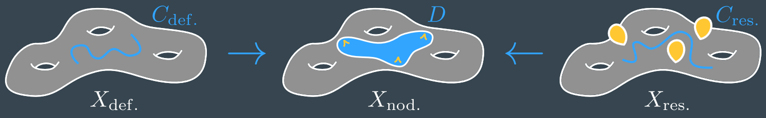
E.g. in our example: $D = \{l_1 = q_1 = 0\} \subset X_{\text{nod.}}$

Curves associated to the transition

Important point:

Such a divisor defines *a pair of curves across the transition*

Namely: curves which limit to this divisor from each side



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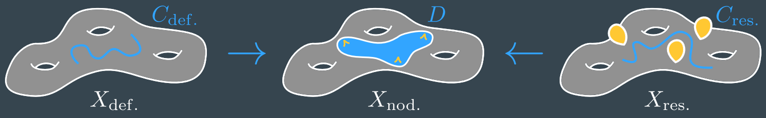

Easiest to describe in example:

$X_{\text{def.}}$	$X_{\text{nod.}}$	$X_{\text{res.}}$
$\cup$	$\cup$	$\cup$
$C_{\text{def.}} = \{l_1 = q_1 = 0\}$	$D = \{l_1 = q_1 = 0\}$	$C_{\text{res.}} = \{x_1 = Q = 0\}$

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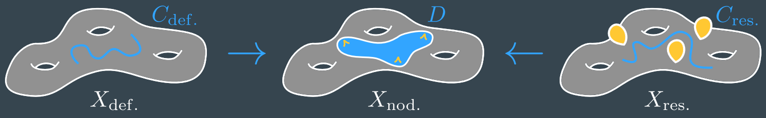


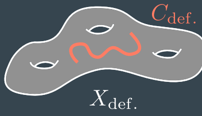
Easiest to describe in example:

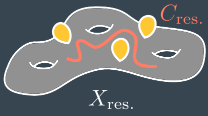
$$X_{\text{def.}} \supset C_{\text{def.}} = \{l_1 = q_1 = 0\} \qquad X_{\text{nod.}} \supset D = \{l_1 = q_1 = 0\} \qquad X_{\text{res.}} \supset C_{\text{res.}} = \{x_1 = Q = 0\}$$

And $C_{\text{def.}} \cong C_{\text{res.}} \cong \left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 0 & 1 & 1 \\ \mathbb{P}^4 & 0 & 5 & 1 & 4 \end{array}\right]$ (inter. in joint ambient space)

Dual 5-brane theories

Consider wrapping 5-branes on $C_{\mathrm{def.}} \subset X_{\mathrm{def.}}$ and $C_{\mathrm{res.}} \subset X_{\mathrm{res.}}$





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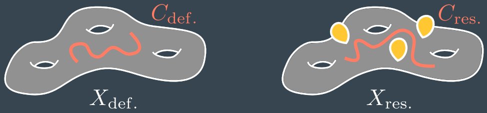

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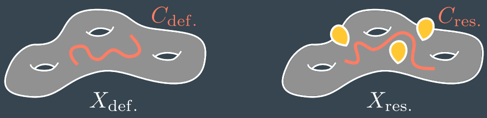

One can prove in generality that for these two theories:

- The anomaly will be cancelled on both sides (with spectator)
- The moduli will match on both sides

So these are dual 5-brane theories

The moduli matching

In (simplest case of) conifold $X_{\mathrm{def.}} \to X_{\mathrm{res.}}$,

$$h^{1,1}(X_{\mathrm{res.}}) + h^{2,1}(X_{\mathrm{res.}}) = (h^{1,1}(X_{\mathrm{def.}}) + 1) + (h^{2,1}(X_{\mathrm{def.}}) - \#\mathbb{P}^1\mathrm{s} + 1)$$
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Change needs to be balanced by difference in brane moduli,

$$h^0(C_{\text{res.}}, \mathcal{N}_{C_{\text{res.}}}) - h^0(C_{\text{def.}}, \mathcal{N}_{C_{\text{def.}}}) + 2 - \#\mathbb{P}^1\text{s} \stackrel{?}{=} 0$$

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Lift computations to D, find: (where $\text{eqs}(Y) = \text{eqs}(X_{\text{nod.}}) - (\,{}^{\text{nodal}}_{\text{eqn}}\,))$

$$h^0(C_{\text{res.}}, \mathcal{N}_{C_{\text{res.}}}) = \text{ind}(\det(\mathcal{N}_{D/Y}))\,, \quad h^0(C_{\text{def.}}, \mathcal{N}_{C_{\text{def.}}}) = \text{ind}(\mathcal{N}_{D/Y})$$

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And taking indices on twist of Koszul resolution

$$0 \to \det(\mathcal{N}_D^\vee) \otimes K_D \to \mathcal{N}_D^\vee \otimes K_D \to K_D \to \mathcal{O}_{D.D} \to 0$$

shows precisely the required relation (using $D \cdot D = \#\mathbb{P}^1\text{s}$)

So the moduli indeed match (in remarkable non-trivial way)

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Natural to ask: Do these theories connect through transition?

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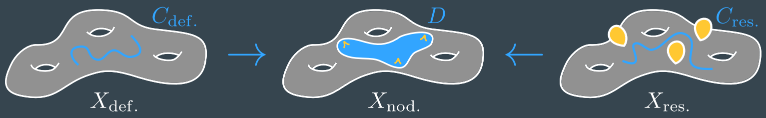

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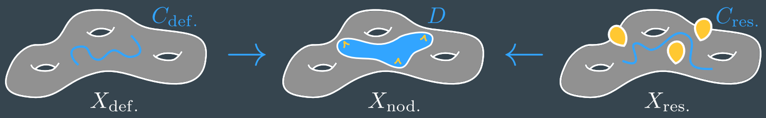

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And duality then explained by symmetry in deforming away

E.g. in our example: both sides deformed by quintic polynomial
- controlling geometry in $X_{\mathrm{def.}}$ and 5-brane in $X_{\mathrm{res.}}$

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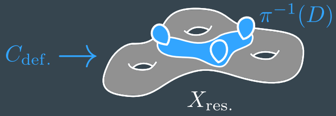

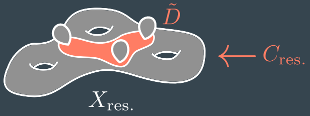

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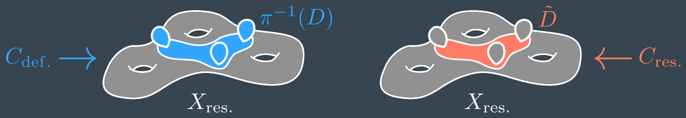

Difference between $\tilde{D}$ and $\pi^{-1}(D)$ is captured by:

$$0 \to \mathcal{O}_{\mathbb{P}^1{}_s}(-2) \to \mathcal{O}_{\pi^{-1}(D)} \otimes \mathcal{O}_{X_{\text{res.}}}(\tilde{D}) \to \mathcal{O}_{\tilde{D}} \to 0$$

(Twist of $\mathcal{O}_{\pi^{-1}(D)}$ by $\mathcal{O}_{X_{\text{res.}}}(\tilde{D})$ is subtle, but: disappears upon deformation to $X_{\text{def.}}$)

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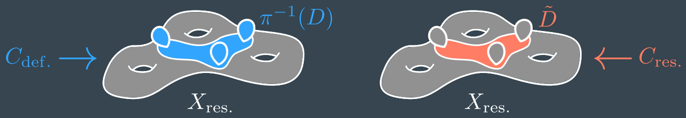

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So seem to need extra 5-brane $\mathcal{O}_{\mathbb{P}^1{}_{\mathrm{s}}}(-2)$ on $X_{\mathrm{res.}}$
for theory on $X_{\mathrm{res.}}$ to meet theory coming from $X_{\mathrm{def.}}$ . . .

Gravitational small instanton transition

But: recall $\mathcal{O}_{\mathbb{P}^1{}_s}(-2)$ is exactly what's needed in gravitational sector (cotangent bundle) to complete transition $X_{\text{res.}} \to X_{\text{def.}}$,

$$0 \to \pi^* \Omega_{\text{nod.}} \to \Omega_{\text{res.}} \to \mathcal{O}_{\mathbb{P}^1{}_s}(-2) \to 0$$

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So our duality of 5-brane theories suggests a process of pair creation of '5-branes': one gauge and one gravitational which performs the transition in an anomaly-consistent way

(This process also seems to naturally underlie target space duality - see James's talk)

Absorption back into bundles

So far discussed 5-branes, but finally absorb (into a spectator bundle V_0 [Candelas et al. 0706.3134]) via small instanton transitions,

$$0 \to V_{\text{def.}} \to V_0 \oplus \mathcal{O}_{X_{\text{def.}}} \to \mathcal{O}_{C_{\text{def.}}} \to 0$$
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- Moduli matching preserved since for Hecke transform:

$$\text{Ext}^1(V,V) = H^1(V_0 \otimes V_0^\vee) \oplus \text{Ext}^1(V_0, \mathcal{I}_C) \oplus \text{Ext}^1(\mathcal{I}_C, V_0) \oplus H^0(C, \mathcal{N}_C)$$

(Change in moduli to full bundle is understood)

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So establish duality of bundle theories (precisely those in TSD) and (claimed) description of a transition between them

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- Procedure to carry heterotic gauge bundle through topological transitions preserving anomaly cancellation
- General method to construct dual heterotic 5-brane theories on CYs with different topologies
- Part of moduli space of heterotic theories on higher $h^{1,1}$ CYs given by theories on lower $h^{1,1}$ CYs